1) Find all \( x \in \mathbb{Z} \), \( 0 \leq x < 77 \), satisfying the congruence \( x^2 + x - 6 \equiv 0 \pmod{77} \).

**Solution:** Since \( x^2 + x - 6 = (x - 2)(x + 3) \), for any prime \( p \),
\( x^2 + x - 6 \equiv 0 \pmod{p} \) \( \implies \) \( x - 2 \equiv 0 \pmod{p} \) or \( x + 3 \equiv 0 \pmod{p} \) \( \implies \)
\( x \equiv 2 \) or \( -3 \pmod{p} \).

So \( x^2 + x - 6 \equiv 0 \pmod{77} \) \( \implies \) \( x^2 + x - 6 \equiv 0 \pmod{7} \) and
\( x^2 + x - 6 \equiv 0 \pmod{11} \) \( \implies \) (\( x \equiv 2 \) or \( -3 \pmod{7} \)) and (\( x \equiv 2 \) or \( -3 \pmod{11} \)).

Hence by CRT, the solutions are \( x \equiv 2, 30, 46, 74 \pmod{77} \).

2) Find all \( k \in \mathbb{Z} \), \( 0 < k < 19 \), which are primitive roots modulo 19.

**Solution:** First, try 2. \( 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 16, 2^5 \equiv 13, 2^6 \equiv 7, 2^7 \equiv 14, 2^8 \equiv 9, 2^9 \equiv 18 \equiv -1, 2^{10} \equiv -2 \equiv 17, 2^{11} \equiv -4 \equiv 15, 2^{12} \equiv -8 \equiv 11, 2^{13} \equiv -16 \equiv 3, 2^{14} \equiv -13 \equiv 6, 2^{15} \equiv -7 \equiv 12, 2^{16} \equiv -14 \equiv 5, 2^{17} \equiv -9 \equiv 10, 2^{18} \equiv 1 \pmod{19} \). Hence 2 is a primitive root modulo 19.

Now the other primitive roots will be \( 2^i \) with \( \gcd(i, 18) = 1 \). Hence
\( 2, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17} \pmod{19} \), or \( 2, 13, 14, 15, 3, 10 \) is the complete list of noncongruent primitive roots modulo 19.

3) Let \( p > 3 \) be a prime, \( S = \{ k \in \mathbb{Z} : \left( \frac{k}{p} \right) = 1, 0 < k < p \} \), and
\( N_p = \sum_{k \in S} k \).

a) Show that \( N_p \equiv 0 \pmod{p} \).

b) Find \( N_p \) if \( p \equiv 1 \pmod{4} \).
Solutions: a) By the sum of squares formula and the fact that 

\[ p > 5, \]

\[ N_p \equiv \sum_{k \in S} k \equiv \sum_{i=1}^{(p-1)/2} i^2 \equiv \frac{p-1}{2} \cdot \frac{p+1}{2} \cdot \frac{p}{6} \equiv \frac{p^2 - 1}{24} \cdot p \equiv 0 \pmod{p} \]

b) If \( p \equiv 1 \pmod{4} \), then \(-1\) is a quadratic residue modulo \( p \). Hence 

\[ \left( \frac{p-k}{p} \right) = \left( \frac{-k}{p} \right) = \left( \frac{-1}{p} \right) \cdot \left( \frac{k}{p} \right) = \left( \frac{k}{p} \right) \] 

and \( p-k \) is a quadratic residue modulo \( p \) iff \( k \) is. This means that the set \( T = \{ p-k : k \in S \} \) consists of quadratic residues modulo \( p \). As \( k \in S \implies 0 < p-k < p \) too, \( T = S \). Then:

\[ N_p = \sum_{k \in S} k = \sum_{l \in T} (p-k) = \sum_{k \in S} p - \sum_{k \in S} k = \frac{p(p-1)}{2} - N_p \]

It follows that

\[ N_p = \frac{p(p-1)}{4} \]

4) Let

\[ f(n) = \begin{cases} 1 & \text{if } n = m^2, m \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases} \]

and \( \lambda = \mu * f \) where \( \mu \) is the Möbius \( \mu \)-function.

a) Is \( f \) multiplicative? Is \( f \) completely multiplicative?

b) Express \( \lambda(n) \) explicitly in terms of the prime factorization \( n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \) of \( n \in \mathbb{Z}^+ \).

c) Is \( \lambda \) multiplicative? Is \( \lambda \) completely multiplicative?

d) Find the Euler product expansion of \( L_\lambda(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \).

e) Express \( L_\lambda(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \) in terms of the Riemann \( \zeta \)-function.

Solution: a) \( f \) is multiplicative: If \( n = ab \) and \( \text{gcd}(a,b) = 1 \), then \( n \) is a perfect square iff both \( a \) and \( b \) are. \( f \) is not completely multiplicative: \( f(5) = 0 \), but \( f(5^2) = 1 \neq 0 = f(5)^2 \).
b) \( \lambda(n) = \sum_{d \mid n} \mu(n/d)f(d) \). \( \mu(n/d) \) is nonzero only when \( n/d \) is square free and \( f(d) \) is nonzero only when \( d \) is a perfect square. If \( n = p_1^{k_1}p_2^{k_2} \cdots p_r^{k_r} \), then the product \( \mu(n/d)f(d) \) is nonzero only when \( d = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r} \) with \( e_i = 2[k_i/2] \). In this case \( n/d = \prod_{k_i \text{ odd}} p_i \) and \( \lambda(n) = (-1)^m \) where \( m \) is the number of odd \( k_i \). Equivalently, \( \lambda(n) = (-1)^{k_1+k_2+\cdots+k_r} \).

c) Since \( \mu \) and \( f \) are multiplicative, \( \lambda = \mu \ast f \) is also multiplicative. But more is true: \( \lambda \) is completely multiplicative.

The formula \( \lambda(n) = (-1)^{k_1+k_2+\cdots+k_r} \) remains valid even when some of \( k_i \)s are 0. Hence if \( p_1, p_2, \ldots, p_r \) are distinct primes, \( k_i, l_i, 1 \leq i \leq r \), are nonnegative integers, and \( n = p_1^{k_1}p_2^{k_2} \cdots p_r^{k_r}, m = p_1^{l_1}p_2^{l_2} \cdots p_r^{l_r}, \) then \( \lambda(n)\lambda(m) = (-1)^{\sum k_i}(-1)^{\sum l_i} = (-1)^{\sum (k_i+l_i)} = \lambda(nm) \).

d) Since \( \lambda \) is completely multiplicative,
\[
L_\lambda(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1 - \lambda(p)p^{-s})^{-1} = \prod_p (1 + p^{-s})^{-1}
\]
e) Using the result of (d),
\[
L_\lambda(s) = \prod_p \frac{1}{1 + p^{-s}} = \prod_p \frac{1 - p^{-s}}{1 - p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}
\]
Or
\[
\lambda = \mu \ast f \implies L_\lambda(s) = L_\mu(s)L_f(s) = \frac{1}{\zeta(s)} \cdot \zeta(2s)
\]
as
\[
L_f(s) = \sum_{m=1}^{\infty} \frac{1}{m^{2s}} = \zeta(2s).
\]