MATH 112 FINAL EXAM SOLUTIONS

Q 1. (10+10 points)

(a) Evaluate the integral: \( I = \int \frac{\sqrt{x^2 - 1}}{x^3} \, dx, \) \( x > 1. \)

Solution. We substitute \( x = \sec \theta. \) Since \( x > 1, \) we have \( 0 \leq \theta < \frac{\pi}{2}. \) Then 
\[
dx = \sec \theta \tan \theta \, d\theta.
\]
So
\[
I = \int \tan \theta \sec \theta \tan \theta \, d\theta = \int \tan^2 \theta \, d\theta = \int \sin^2 \theta \, d\theta
\]
\[
= \frac{1 - \cos 2\theta}{2} \, d\theta = \frac{1}{2} \left( \theta - \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{2} \left( \theta - \sin \theta \cos \theta \right) + C
\]

(b) Let \( R \) be the region in the plane bounded above by the parabola \( y = x^2, \) below by the lower half of the circle \((x-2)^2 + y^2 = 4,\) and on the right by the line \( x = 2. \) Assume \( R \) is revolved about the \( y \)-axis, and a solid is generated. Draw the region \( R, \) and set up the integral for the volume of this solid by using the cylindrical shell method. Do not evaluate.

Solution. We use the method of cylindrical shells, and get
\[
V = 2\pi \int_0^2 x \left( x^2 - (\sqrt{4 - (x-2)^2}) \right) \, dx = 2\pi \int_0^2 x \left( x^2 + \sqrt{4 - (x-2)^2} \right) \, dx
\]

Q 2. (10+10 points) Determine whether the following series are convergent or divergent by using a test. Show all your work and write the name of the test that you use.

(a) \( \sum_{n=1}^{\infty} \sin(1/n^2) \sin(n^2). \)

Solution. We have \( a_n = \sin(1/n^2) \sin(n^2). \) We use the following two inequalities:
i) \( |\sin x| \leq |x| \) (useful when \( |x| \) is small), and ii) \( |\sin x| \leq 1 \) (useful when \( |x| \) is large). So
i) Let $f$ conditions of AST: i) Is
\[ \sum |a_n| \leq 1; \]
choose $b_n = \frac{1}{n^2}$. Then since $\sum b_n = \sum \frac{1}{n^2}$ is convergent ($p$-series with $p = 2 > 1$), by DCT, the series $\sum |a_n|$ is convergent, so by ACT the series $\sum a_n = \sum \sin(1/n^2) \sin(n^2)$ is convergent.

ii) $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{n^2 + 3n + 10}$.

**Solution.** This is an alternating series $\sum (-1)^{n+1} u_n$, where $u_n = \frac{2}{n^2 + 3n + 10}$. We check the conditions of AST: i) Is $\{u_n\}$ decreasing?, ii) Is $\lim_{n \to \infty} u_n = 0$?

i) Let $f(x) = \frac{x}{x^2 + 3x + 10}, x > 0$. Then $u_n = f(n)$. We have
\[
\frac{df}{dx} = \frac{-x^2 + 10}{(x^2 + 3x + 10)^2} \Rightarrow f'(x) < 0 \text{ for } x > \sqrt{10}
\Rightarrow f(x) \text{ is decreasing for } x > \sqrt{10}
\Rightarrow u_n \text{ is decreasing for } n \geq 4.
\]

ii) $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{2}{n^2 + \ln(n) + 10} = 0$.

Thus by AST the series $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2}{n^2 + 3n + 10}$ is convergent.

**Q 3. (24 points)** Show all your work.

a) i. Find the interval of convergence of the following power series.

a) ii. For which $x$ in this interval, is the series absolutely convergent?

a) iii. For which $x$ in this interval is this series conditionally convergent?

\[ \sum_{n=0}^{\infty} \frac{(x + 3)^{2n}}{\sqrt{n + 1} \cdot 4^n}. \]

**Solution.** We apply the Ratio Test (or the Root Test) to the series $\sum_{n=0}^{\infty} \left| \frac{(x + 3)^{2n}}{\sqrt{n + 1} \cdot 4^n} \right|$.

\[
\rho = \lim_{n \to \infty} \left| \frac{(x + 3)^{2n+2}}{\sqrt{n + 2} \cdot 4^{n+1}} \right| \left| \frac{\sqrt{n + 1} \cdot 4^n}{(x + 3)^{2n}} \right| = \lim_{n \to \infty} \frac{|x + 3|^2}{4} \frac{\sqrt{n + 1}}{\sqrt{n + 2}} = \frac{|x + 3|^2}{4}.
\]

We have

- series converges absolutely if $\rho < 1 \Leftrightarrow |x + 3|^2 < 4 \Leftrightarrow |x + 3| < 2 \Leftrightarrow -5 < x < -1$
- series diverges if $\rho > 1 \Leftrightarrow |x + 3|^2 > 4 \Leftrightarrow x < -5$ or $x > -1$
- test is inconclusive if $\rho = 1 \Leftrightarrow x = -5$ or $x = -1$.

When $x = -5$ or $x = -1$, we have $x + 3 = \mp 2$, so the series becomes
\[ \sum_{n=0}^{\infty} \frac{(\mp 2)^{2n}}{\sqrt{n + 1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{4^n}{\sqrt{n + 1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n + 1}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}, \]
and this series is divergent ($p$-series with $p = 1/2 \leq 1$).

Thus:

i. Interval of convergence is $(-5, -1)$.

ii. The series converges absolutely for all $x \in (-5, -1)$.

iii. There is no $x$ in the interval for which the series converges conditionally.
b) Let \( f(x) = \sum_{n=0}^{\infty} \frac{(x + 3)^{2n}}{\sqrt{n + 1} \cdot 4^n} \). Find \( f^{(148)}(-3) \) and \( f^{(149)}(-3) \).

**Solution.** Remember if \( f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \), then \( c_n = \frac{f^{(n)}(a)}{n!} \), thus \( f^{(n)}(a) = n! \cdot c_n \).

In this question \( a = -3 \), and \( c_n = 0 \) if \( n \) is odd. More precisely,

\[
\begin{align*}
c_{2n} &= \frac{1}{\sqrt{n + 1} \cdot 4^n}, \\
c_{2n+1} &= 0.
\end{align*}
\]

So

\[
\begin{align*}
f^{(148)}(-3) &= 148! \cdot c_{148} = 148! \cdot \frac{1}{\sqrt{74 + 1} \cdot 4^{74}} = \frac{148!}{\sqrt{75} \cdot 4^{74}}, \\
f^{(149)}(-3) &= 149! \cdot c_{149} = 0.
\end{align*}
\]

**Q 4.** (16 points) Find a polynomial \( P(x) \) of the least degree which approximates the function

\[
F(x) = \int_{0}^{x} e^{-t^2} \, dt
\]

with \( |\text{error}| < 1/1000 \) for all \( x \in [0, 1] \).

**Solution.** We have

\[
e^s = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^n}{n!} + \cdots, \quad \text{all } s \Rightarrow e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \cdots, \quad \text{all } t.
\]

Thus

\[
F(x) = \int_{0}^{x} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \cdots \right) \, dt
\]

\[
= t - \frac{t^3}{3} + \frac{t^5}{2! 5} - \cdots + (-1)^n \frac{x^{2n+1}}{n! (2n + 1)} + \cdots \bigg|_{0}^{x}
\]

\[
= x - \frac{x^3}{3} + \frac{x^5}{2! 5} - \cdots + (-1)^n \frac{x^{2n+1}}{n! (2n + 1)} + \cdots
\]

When \( 0 \leq x \leq 1 \), the last power series is an alternating series with \( u_n = \frac{x^{2n+1}}{n! (2n + 1)} \). For \( 0 \leq x \leq 1 \) we have

\[
\frac{u_{n+1}}{u_n} = \frac{x^{2n+3}}{(n + 1)! (2n + 3)} \cdot \frac{n! (2n + 1)}{x^{2n+1}} = \frac{x^2}{n + 1} \cdot \frac{2n + 1}{2n + 3} \leq 1 \Rightarrow u_{n+1} \leq u_n.
\]

Also,

\[
0 \leq x \leq 1 \Rightarrow 0 \leq u_n \leq \frac{1}{n! (2n + 1)} \Rightarrow \lim_{n \to \infty} u_n = 0.
\]

Thus we can use the AST error formula, that is if we stop at the \( n \)-th term \((-1)^n \frac{x^{2n+1}}{n! (2n + 1)}\) then the absolute error will be less than the absolute value of the first unused term. That is \( |\text{error}| < \frac{x^{2n+3}}{(n + 1)! (2n + 3)} \). Thus

\[
|\text{error}| < \frac{x^{2n+3}}{(n + 1)! (2n + 3)} \leq \frac{1}{(n + 1)! (2n + 3)}.
\]
Now we choose the smallest positive integer \( n \) such that \( \frac{1}{(n+1)!(2n+3)} \leq \frac{1}{1000} \), that is \( 1000 \leq (n + 1)! (2n + 3) \). We see that for \( n = 4 \), \((n + 1)! (2n + 3) = 1320\) and for \( n = 3 \), \((n + 1)! (2n + 3) = 254\). Thus \( n = 4 \) is good. So we take
\[
P(x) = x - \frac{x^3}{3} + \frac{x^5}{2! 5} - \frac{x^7}{3! 7} + \frac{x^9}{4! 9},
\]
that is the last term is \((-1)^n \frac{x^{2n+1}}{n!(2n+1)}\) with \( n = 4 \).

**Q 5. (20 points)** Given the point \( P_0(0, -3, 1) \), and the line \( L_1: x = 5t - 3, y = -2t + 2, z = 4t - 4, -\infty < t < \infty \),

(a) Find the equation of the plane \( M \) which passes through the point \( P_0 \) and is perpendicular to the line \( L_1 \).

**Solution.** We have \( n_M = v_{L_1} = 5i - 2j + 4k \). So
\[
M : 5(x - 0) - 2(y + 3) + 4(z - 1) = 0 \text{ or } M : 5x - 2y + 4z = 10.
\]

(b) Find the coordinates of the intersection point \( Q_0 \) of the line \( L_1 \) and the plane \( M \) in part (a).

**Solution.** Let \( Q_0 \) have coordinates \((x, y, z)\). Then
\[
\begin{align*}
Q_0 \text{ is on } L_1 & \Rightarrow x = 5t - 3, y = -2t + 2, z = 4t - 4, \\
Q_0 \text{ is on } M & \Rightarrow 5x - 2y + 4z = 10,
\end{align*}
\]
so
\[
5(5t - 3) - 2(-2t + 2) + 4(4t - 4) = 10 \Rightarrow 45t = 45 \Rightarrow t = 1.
\]
So \( x = 2, y = 0, z = 0 \), that is, the intersection point is \( Q_0(2, 0, 0) \).

(c) Find the parametric equations of the line \( L \) that passes through the point \( P_0 \) and intersects the line \( L_1 \) orthogonally.

**Hint.** Use part (b).

**Solution.** The direction vector of the line \( L \) is \( v_L = P_0 \overrightarrow{Q_0} = 2i + 3j - k \). Taking the point on the line \( L \) as \( P_0(0, -3, 1) \), we get
\[
L : x = 2t, y = 3t - 3, z = -t + 1, -\infty < t < \infty.
\]