

4. Find and classify the critical points of $f(x, y) = y^2 - c^2y + cx^2y$ for each value of the constant c .

$$\left. \begin{aligned} \textcircled{i} \quad f_x &= 2cxy = 0 \\ \textcircled{ii} \quad f_y &= 2y - c^2 + cx^2 = 0 \end{aligned} \right\}$$

If $c \neq 0$, then: $\textcircled{i} \Rightarrow x=0$ or $y=0$

$$x=0 \text{ and } \textcircled{ii} \Rightarrow y = \frac{c^2}{2} \Rightarrow (x, y) = (0, \frac{c^2}{2})$$

$$y=0 \text{ and } \textcircled{ii} \Rightarrow x = \pm\sqrt{c} \text{ if } c > 0 \Rightarrow (x, y) = (\pm\sqrt{c}, 0) \text{ if } c > 0$$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2cy & 2cx \\ 2cx & 2 \end{vmatrix}$$

$$\Delta(0, \frac{c^2}{2}) = \begin{vmatrix} c^3 & 0 \\ 0 & 2 \end{vmatrix} = 2c^3$$

If $c < 0$, then $\Delta(0, \frac{c^2}{2}) = 2c^3 < 0 \Rightarrow f$ has a saddle point at $(0, \frac{c^2}{2})$

$$\text{If } c > 0, \text{ then } \Delta(0, \frac{c^2}{2}) = 2c^3 > 0 \text{ and } f_{yy}(0, \frac{c^2}{2}) = 2 > 0$$

$\Rightarrow f$ has a local minimum at $(0, \frac{c^2}{2})$

$$\Delta(\pm\sqrt{c}, 0) = \begin{vmatrix} 0 & \pm 2c\sqrt{c} \\ \pm 2c\sqrt{c} & 0 \end{vmatrix} = -4c^3 < 0 \text{ if } c > 0$$

$\Rightarrow f$ has saddle points at $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$ if $c > 0$

If $c=0$, then:

$$\left. \begin{aligned} f_x &= 0 \\ f_y &= 2y \end{aligned} \right\} \Rightarrow \text{Every point } (a, 0) \text{ on the } x\text{-axis is a critical point.}$$

$$f(a, 0) = 0 \leq y^2 = f(x, y) \text{ for all } (x, y)$$

$\Rightarrow f$ has a local minimum at every point $(a, 0)$ on the x -axis.