

1. $f(x, y)$ is a *harmonic function* on an open set D if it satisfies the equation

$$f_{xx} + f_{yy} = 0$$

for all (x, y) in D .

Harmonic functions appear and play important roles in various areas of mathematics, science, and engineering—from electromagnetism to stochastic processes.

Determine all values of the constant c for which

$$f(x, y) = xy(x^2 + y^2)^c$$

is a harmonic function on $D = \{(x, y) : (x, y) \neq (0, 0)\}$.

$$f_x = y(x^2 + y^2)^c + xy \cdot c(x^2 + y^2)^{c-1} \cdot 2x = y(x^2 + y^2)^c + 2cx^2y(x^2 + y^2)^{c-1}$$

$$\begin{aligned} f_{xx} &= y \cdot c(x^2 + y^2)^{c-1} \cdot 2x + 2c \cdot 2xy(x^2 + y^2)^{c-1} + 2cx^2y \cdot (c-1)(x^2 + y^2)^{c-2} \cdot 2x \\ &= 6cxy(x^2 + y^2)^{c-1} + 4c(c-1) \cdot x^3y(x^2 + y^2)^{c-2} \end{aligned}$$

Similarly,

$$f_{yy} = 6cxy(x^2 + y^2)^{c-1} + 4c(c-1) \cdot xy^3(x^2 + y^2)^{c-2}$$

Hence:

$$\begin{aligned} f_{xx} + f_{yy} &= 12cxy(x^2 + y^2)^{c-1} + 4c(c-1) \cdot xy \cdot (x^2 + y^2) \cdot (x^2 + y^2)^{c-2} \\ &= 4c(c+2)xy(x^2 + y^2)^{c-1} \end{aligned}$$

Therefore:

$$f_{xx} + f_{yy} = 0 \text{ for all } (x, y) \neq (0, 0) \iff c(c+2) = 0 \iff c = 0 \text{ or } c = -2$$