

4. Find and classify the critical points of $f(x, y) = x^5 + y^2 - x^3y - x^2y$.

$$\textcircled{1} f_x = 5x^4 - 3x^2y - 2xy = 0$$

$$\textcircled{2} f_y = 2y - x^3 - x^2 = 0 \Rightarrow y = \frac{x^3 + x^2}{2}$$

$$\Rightarrow 5x^4 - (3x^2 + 2x) \cdot \frac{1}{2}(x^3 + x^2) = 0$$

$$\Rightarrow x^3 \cdot (3x^2 - 5x + 2) = 0 \Rightarrow x = 0, \quad x = 1 \quad \text{or} \quad x = \frac{2}{3}$$

$\Downarrow \textcircled{2} \quad y = 0$
 $\Downarrow \textcircled{2} \quad y = 1$
 $\Downarrow \textcircled{2} \quad y = \frac{10}{27}$

Critical points are $(x, y) = (0, 0), (1, 1), (\frac{2}{3}, \frac{10}{27})$.

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 20x^3 - 6xy - 2y & -3x^2 - 2x \\ -3x^2 - 2x & 2 \end{vmatrix}$$

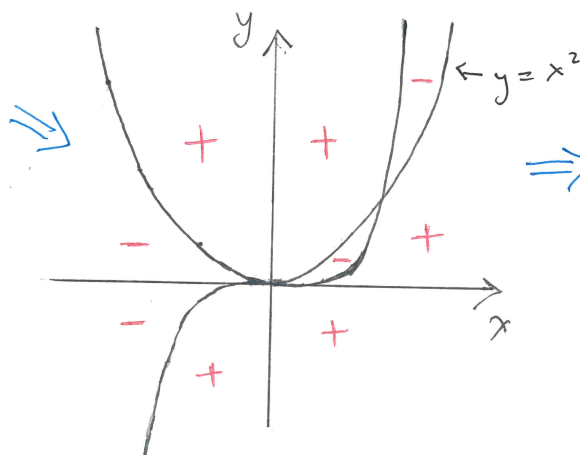
$$\Delta(1, 1) = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} = 12 \cdot 2 - (-5)^2 = -1 < 0 \Rightarrow (1, 1) \text{ is a saddle point.}$$

$$\Delta\left(\frac{2}{3}, \frac{10}{27}\right) = \begin{vmatrix} 100/27 & -8/3 \\ -8/3 & 2 \end{vmatrix} = \frac{100}{27} \cdot 2 - \left(-\frac{8}{3}\right)^2 = \frac{8}{27} > 0$$

$\Rightarrow (\frac{2}{3}, \frac{10}{27})$ is a local minimum

and $f_{yy}(\frac{2}{3}, \frac{10}{27}) = 2 > 0$

$$f(x, y) = (x^3 - y)(x^2 - y)$$



$\Rightarrow (0, 0)$ is a saddle point.

$$\left. \begin{array}{l} \Delta > 0 \text{ and } f_{xx} > 0 \Rightarrow \text{local minimum} \\ \Delta > 0 \text{ and } f_{xx} < 0 \Rightarrow \text{local maximum} \\ \Delta < 0 \Rightarrow \text{saddle point} \end{array} \right\} \text{ where } \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$