

3. Consider the surface $\mathcal{S} : \overbrace{xy^2 + yz^2 + zx^2}^{f(x,y,z)} = -1$.

a. Find an equation for the tangent plane $\mathcal{P}_1$ to the surface $\mathcal{S}$ at the point $P_1(1, -1, -1)$.

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (y^2 + 2xz) \vec{i} + (2xy + z^2) \vec{j} + (2yz + x^2) \vec{k}$$

$$\begin{aligned} \vec{\nabla} f(P_1) &= ((-1)^2 + 2 \cdot 1 \cdot (-1)) \vec{i} + (2 \cdot 1 \cdot (-1) + (-1)^2) \vec{j} + (2 \cdot (-1) \cdot (-1) + 1^2) \vec{k} \\ &= -\vec{i} - \vec{j} + 3\vec{k} \end{aligned}$$

$$\mathcal{P}_1: (-1) \cdot (x-1) + (-1) \cdot (y-(-1)) + 3 \cdot (z-(-1)) = 0$$

b. Find an equation for the tangent plane $\mathcal{P}_2$ to the surface $\mathcal{S}$ at the point $P_2(1, -1, 2)$.

$$\begin{aligned} \vec{\nabla} f(P_2) &= ((-1)^2 + 2 \cdot 1 \cdot 2) \vec{i} + (2 \cdot 1 \cdot (-1) + 2^2) \vec{j} + (2 \cdot (-1) \cdot 2 + 1^2) \vec{k} \\ &= 5\vec{i} + 2\vec{j} - 3\vec{k} \end{aligned}$$

$$\mathcal{P}_2: 5 \cdot (x-1) + 2 \cdot (y-(-1)) + (-3) \cdot (z-2) = 0$$

c. Find parametric equations for the line of intersection $L = \mathcal{P}_1 \cap \mathcal{P}_2$ of the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

$$\left. \begin{aligned} \mathcal{P}_1: x + y - 3z &= 3 \\ \mathcal{P}_2: 5x + 2y - 3z &= -3 \end{aligned} \right\}$$

$$\text{Let } z = t. \text{ Then: } \begin{aligned} x + y &= 3t + 3 & \textcircled{1} \\ 5x + 2y &= 3t - 3 & \textcircled{2} \end{aligned}$$

$$-2 \times \textcircled{1} + \textcircled{2} \Rightarrow 3x = -3t - 9 \Rightarrow x = -t - 3 \quad \rightarrow y = 4t + 6$$

$$\text{Hence, } L: \begin{cases} x = -t - 3 \\ y = 4t + 6 \\ z = t \end{cases} \quad (-\infty < t < \infty)$$