

2. Consider the function $f(x, y, z) = \frac{1}{3}x^3 - x^2y + 3xy^2 - \frac{1}{3}z^3$.

a. Compute ∇f .

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (x^2 - 2xy + 3y^2) \vec{i} + (-x^2 + 6xy) \vec{j} - z^2 \vec{k}$$

b. Find the maximum rate of change of f at the point $P_0(3, 1, \sqrt{2})$ and find the unit vector $\mathbf{u}$ along which this maximum rate of change occurs.

$$\vec{\nabla} f(P_0) = (3^2 - 2 \cdot 3 \cdot 1 + 3 \cdot 1^2) \vec{i} + (-3^2 + 6 \cdot 3 \cdot 1) \vec{j} - (\sqrt{2})^2 \vec{k} = 6 \vec{i} + 9 \vec{j} - 2 \vec{k}$$

The maximum rate of change is $|\vec{\nabla} f(P_0)| = \sqrt{6^2 + 9^2 + (-2)^2} = \sqrt{121} = 11$

and this occurs along $\vec{u} = \frac{\vec{\nabla} f(P_0)}{|\vec{\nabla} f(P_0)|} = \frac{6 \vec{i} + 9 \vec{j} - 2 \vec{k}}{11}$

c. Let $\mathcal{S}$ be the set of points at which the rate of change of f in the direction of the vector $\mathbf{A} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ is 0. Show that $\mathcal{S}$ is the union $\mathcal{P}_1 \cup \mathcal{P}_2$ of two planes $\mathcal{P}_1$ and $\mathcal{P}_2$, and find an equation for each of these planes.

$$\vec{\nabla} f \cdot \vec{A} = 0 \iff 3 \cdot (x^2 - 2xy + 3y^2) + (-1) \cdot (-x^2 + 6xy) + 1 \cdot (-z^2) = 0$$

$$\iff 4x^2 - 12xy + 9y^2 - z^2 = 0$$

$$\iff (2x - 3y)^2 - z^2 = 0 \iff (2x - 3y - z) \cdot (2x - 3y + z) = 0$$

$$\iff 2x - 3y - z = 0 \text{ or } 2x - 3y + z = 0$$

Hence, $\mathcal{P}_1: 2x - 3y - z = 0$ and $\mathcal{P}_2: 2x - 3y + z = 0$.