

4a. Determine the interval of convergence of the power series $\sum_{n=0}^{\infty} \underbrace{\sin(\pi/2^{n+1})}_{a_n} x^n$.

$$L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{\left| \sin\left(\frac{\pi}{2^{n+2}}\right) \cdot x^{n+1} \right|}{\left| \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot x^n \right|} = \lim_{n \rightarrow \infty} \frac{\sin(\pi/2^{n+2}) / (\pi/2^{n+2})}{\sin(\pi/2^{n+1}) / (\pi/2^{n+1})} \cdot \frac{|x|}{2}$$

$$= \frac{1}{1} \cdot \frac{|x|}{2} = \frac{|x|}{2} \quad \text{as} \quad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

If $|x| < 2$, then $L < 1$ and the power series converges absolutely by R.T. }
 If $|x| > 2$, then $L > 1$ and the power series diverges by R.T.

$$\Rightarrow R = 2$$

If $x = 2$, then: $\sum_{n=0}^{\infty} \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot 2^n = \sum_{n=0}^{\infty} \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot 2^n$

Since $\lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot 2^n = \lim_{n \rightarrow \infty} \frac{\sin(\pi/2^{n+1})}{\pi/2^{n+1}} \cdot \frac{\pi}{2} = 1 \cdot \frac{\pi}{2} = \frac{\pi}{2} \neq 0$,

the power series diverges at $x = 2$ by nTT.

If $x = -2$, then: $\sum_{n=0}^{\infty} \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot (-2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot 2^n$

Since $\lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot 2^n = \frac{\pi}{2} \neq 0$, $\lim_{n \rightarrow \infty} (-1)^n \sin\left(\frac{\pi}{2^{n+1}}\right) \cdot 2^n$ does not exist, and

the power series diverges at $x = -2$ by nTT.

Hence, the interval of convergence is $I = (-2, 2)$.

4b. Show that $\sum_{n=0}^{\infty} \sin(\pi/2^{n+1}) < 1 + \frac{\pi}{2}$.

$$\sum_{n=0}^{\infty} \sin\left(\frac{\pi}{2^{n+1}}\right) = \sin\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2^{n+1}}\right)$$

as $0 < \sin \theta < \theta$ for $\theta > 0$

$$< 1 + \underbrace{\sum_{n=1}^{\infty} \frac{\pi}{2^{n+1}}}_{\text{geometric series with } a = \frac{\pi}{4} \text{ and } r = \frac{1}{2}} = 1 + \frac{\pi/4}{1 - \frac{1}{2}} = 1 + \frac{\pi}{2}$$

converges as $|r| = |\frac{1}{2}| = \frac{1}{2} < 1$