

2. Determine whether each of the following series converges or diverges.

a. $\sum_{n=1}^{\infty} \underbrace{\frac{n!n^n}{(2n)!}}_{a_n}$

$$L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)! (n+1)^{n+1}}{(2(n+1))!}}{\frac{n! n^n}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n \rightarrow \infty} \frac{n! \cdot (n+1)}{n!} \cdot \frac{(n+1) \cdot (n+1)^n}{n^n} \cdot \frac{(2n)!}{(2n)! (2n+1)(2n+2)}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+1)(2n+2)} \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \frac{1}{4} \cdot e = \frac{e}{4}$$

by "useful limits"

Since $L = \frac{e}{4} < 1$, $\sum_{n=1}^{\infty} \frac{n!n^n}{(2n)!}$ converges by R.T.

b. $\sum_{n=1}^{\infty} (\ln(n+1) - \ln(n))^2$

$$C = \lim_{n \rightarrow \infty} \frac{(\ln(n+1) - \ln(n))^2}{\frac{1}{n^2}} = \left(\lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \right)^2 = \left(\lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots}{\frac{1}{n}} \right)^2$$

$$= \left(\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots\right) \right)^2 = 1^2 = 1$$

Since $C < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p -series with $p=2 > 1$),

$\sum_{n=1}^{\infty} (\ln(n+1) - \ln(n))^2$ converges by L.C.T.