

1. A sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_1 = 0$, $a_2 = 1$, and $a_n = \frac{1+a_{n-2}}{1+a_{n-1}}$ for $n \geq 3$.

a. Fill in the boxes. No explanation is required. No partial points will be given.

$$a_3 = \boxed{\frac{1}{2}} \quad a_4 = \boxed{\frac{4}{3}} \quad a_5 = \boxed{\frac{9}{14}} \quad a_6 = \boxed{\frac{98}{69}}$$

b. Show that $0 < a_n < \frac{1}{a_{n+1}}$ for all $n \geq 2$.

* For $n=2$, $a_2=1$, $a_3=\frac{1}{2}$, and $0 < 1 < 2 = \frac{1}{1/2}$.

* Assume that $0 < a_k < \frac{1}{a_{k+1}}$ for some $k \geq 2$. Then

$$0 < a_k < \frac{1}{a_{k+1}} \Rightarrow 0 < 1 + a_k < \frac{a_{k+1} + 1}{a_{k+1}} \Rightarrow 0 < a_{k+1} < \frac{1 + a_{k+1}}{1 + a_k} = \frac{1}{a_{k+2}}$$

Hence, by induction, $0 < a_n < \frac{1}{a_{n+1}}$ for all $n \geq 2$.

c. Use Part b to show that $a_n < a_{n+2}$ for all $n \geq 2$.

$$a_{n+2} - a_n = \frac{1 + a_n}{1 + a_{n+1}} - a_n = \frac{1 - a_n a_{n+1}}{1 + a_{n+1}} > 0 \text{ for } n \geq 2$$

as $a_n a_{n+1} < 1$ for $n \geq 2$
by Part (b).

d. Show that $\lim_{n \rightarrow \infty} a_{2n}$ exists.

By Parts (c) and (b), $\{a_{2n+1}\}_{n=1}^{\infty}$ is an increasing sequence with positive terms. Hence $\{\frac{1}{a_{2n+1}}\}_{n=1}^{\infty}$ is a decreasing sequence. Therefore,

$$2 = \frac{1}{a_3} \geq \frac{1}{a_{2n+1}} > a_{2n} \text{ for all } n \geq 1 \text{ by Part (b).}$$

Hence $\{a_{2n}\}_{n=1}^{\infty}$ is bounded above. $\Rightarrow \lim_{n \rightarrow \infty} a_{2n}$ exists

By Part (c), $\{a_{2n}\}_{n=1}^{\infty}$ is increasing. $\Rightarrow$ by M.S.T.