

2a. Evaluate the double integral $\iint_{D_1} (x^2 - y^2)^2 (x^2 + y^2) dA$ where $D_1 = \{(x, y) : x^2 + y^2 \leq 1\}$.

$$\begin{aligned}
 \iint_{D_1} (x^2 - y^2)^2 (x^2 + y^2) dA &= \int_0^{2\pi} \int_0^1 ((r\cos\theta)^2 - (r\sin\theta)^2)^2 r^2 \cdot r dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 \cos^2 2\theta \cdot r^7 dr d\theta = \int_0^{2\pi} \cos^2 2\theta \cdot \left[\frac{r^8}{8} \right]_{r=0}^{r=1} d\theta \\
 &= \frac{1}{8} \int_0^{2\pi} \frac{1 + \cos 4\theta}{2} d\theta = \frac{1}{8} \left[\frac{1}{2} \theta + \frac{1}{8} \sin 4\theta \right]_0^{2\pi} \\
 &= \frac{1}{8} \left(\frac{1}{2} \cdot 2\pi + \frac{1}{8} \sin 8\pi - \frac{1}{2} \cdot 0 - \frac{1}{8} \sin 0 \right) = \frac{\pi}{8}
 \end{aligned}$$

2b. Find the area of the region $D_2 = \{(x, y) : (3x - 2y)^2 + (x + 4y)^2 \leq 7\}$.

Let $u = 3x - 2y$, $v = x + 4y$, and $G = \{(u, v) : u^2 + v^2 \leq (\sqrt{7})^2\}$.

$$\begin{aligned}
 (\text{Area of } D_2) &= \iint_{D_2} dx dy = \iint_G \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \frac{1}{14} \iint_G du dv \\
 &= \frac{1}{14} (\text{Area of } G) = \frac{1}{14} \cdot \pi \cdot (\sqrt{7})^2 = \frac{\pi}{2}
 \end{aligned}$$

where we used:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 3 \cdot 4 - 1 \cdot (-2) = 14$$

$$\Rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}} = \frac{1}{14}$$

$$\iint_D f(x, y) dx dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \quad \text{where} \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$