

4. Find and classify the critical points of the function $f(x, y) = 4x + 2xy - 2xy^2 - 3x^2$.

$$\begin{aligned} \textcircled{1} \quad f_x &= 4 + 2y - 2y^2 - 6x = 0 \\ \textcircled{2} \quad f_y &= 2x - 4xy = 0 \end{aligned} \quad \Rightarrow \quad 2x(1-2y) = 0 \Rightarrow x=0 \text{ or } y=\frac{1}{2}$$

$$\textcircled{1} \text{ and } x=0 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y=2 \text{ or } y=-1$$

$$\Downarrow$$

$$(x, y) = (0, 2), (0, -1)$$

$$\textcircled{1} \text{ and } y=\frac{1}{2} \Rightarrow \frac{9}{2} - 6x = 0 \Rightarrow x = \frac{3}{4}$$

$$\Downarrow$$

$$(x, y) = \left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6 & 2-4y \\ 2-4y & -4x \end{vmatrix}$$

$$\Delta(0, 2) = \begin{vmatrix} -6 & -6 \\ -6 & 0 \end{vmatrix} = (-6) \cdot 0 - (-6)^2 = -36 < 0 \Rightarrow (0, 2) \text{ is a saddle point}$$

$$\Delta(0, -1) = \begin{vmatrix} -6 & 6 \\ 6 & 0 \end{vmatrix} = (-6) \cdot 0 - 6^2 = -36 < 0 \Rightarrow (0, -1) \text{ is a saddle point}$$

$$\Delta\left(\frac{3}{4}, \frac{1}{2}\right) = \begin{vmatrix} -6 & 0 \\ 0 & -3 \end{vmatrix} = (-6) \cdot (-3) - 0^2 = 18 > 0$$

$$\left. \begin{aligned} &\text{and } f_{xx}\left(\frac{3}{4}, \frac{1}{2}\right) = -6 < 0 \end{aligned} \right\} \Rightarrow \left(\frac{3}{4}, \frac{1}{2}\right) \text{ is a local max}$$

$$\left. \begin{aligned} \Delta > 0 \text{ and } f_{xx} > 0 &\Rightarrow \text{local minimum} \\ \Delta > 0 \text{ and } f_{xx} < 0 &\Rightarrow \text{local maximum} \\ \Delta < 0 &\Rightarrow \text{saddle point} \end{aligned} \right\} \text{ where } \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$
