

3. Suppose $f(x, y, z)$ is a differentiable function with $\nabla f(P_0) = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ at the point $P_0(2, 1, 1/2)$.

[Do not assume anything about the function f beyond what is given in the sentence above.]

a. Find the directional derivative of f at P_0 in the direction of the vector $\mathbf{A} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

$$\text{The directional derivative is } D_{\vec{u}}f(P_0) = \vec{\nabla}f(P_0) \cdot \vec{u} = (3\vec{i} - \vec{j} - 2\vec{k}) \cdot \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\text{where } \vec{u} = (\text{the direction of } \vec{A}) = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}$$

b. Find $D_{\vec{u}}f(P_0)$ if $\vec{u}$ is a unit vector which makes an angle of 120° with $\nabla f(P_0)$.

$$D_{\vec{u}}f(P_0) = \vec{\nabla}f(P_0) \cdot \vec{u} = \underbrace{|\vec{\nabla}f(P_0)|}_{\sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}} \underbrace{|\vec{u}|}_{1} \underbrace{\cos \theta}_{\cos 120^\circ = -\frac{1}{2}} = \sqrt{14} \cdot 1 \cdot \left(-\frac{1}{2}\right) = -\sqrt{\frac{7}{2}}$$

c. Find an equation for the tangent plane to the level surface of f passing through the point P_0 .

$$3 \cdot (x-2) + (-1) \cdot (y-1) + (-2) \cdot (z - \frac{1}{2}) = 0$$

(or)

$$3x - y - 2z = 4$$

d. Find $\nabla g(2, 1/2)$ if $g(x, y) = f(x^2y, xy, xy^2)$.

$$g_x = f_1 \cdot 2xy + f_2 \cdot y + f_3 \cdot y^2$$

$$\Rightarrow g_x(2, \frac{1}{2}) = f_1(2, 1, \frac{1}{2}) \cdot 2 + f_2(2, 1, \frac{1}{2}) \cdot \frac{1}{2} + f_3(2, 1, \frac{1}{2}) \cdot \frac{1}{4} \\ = 3 \cdot 2 + (-1) \cdot \frac{1}{2} + (-2) \cdot \frac{1}{4} = 5$$

$$g_y = f_1 \cdot x^2 + f_2 \cdot x + f_3 \cdot 2xy$$

$$\Rightarrow g_y(2, \frac{1}{2}) = f_1(2, 1, \frac{1}{2}) \cdot 4 + f_2(2, 1, \frac{1}{2}) \cdot 2 + f_3(2, 1, \frac{1}{2}) \cdot 2 \\ = 3 \cdot 4 + (-1) \cdot 2 + (-2) \cdot 2 = 6$$

$$\text{Hence, } \vec{\nabla}g(2, \frac{1}{2}) = 5\vec{i} + 6\vec{j}$$