

4. A *Carnot cycle* is a thermodynamic process consisting of consecutive isothermal and isentropic expansions and compressions modeling an ideal heat engine (or refrigerator).

The work done by the engine (or on the refrigerator) is equal to the area A of the region D enclosed by the curves

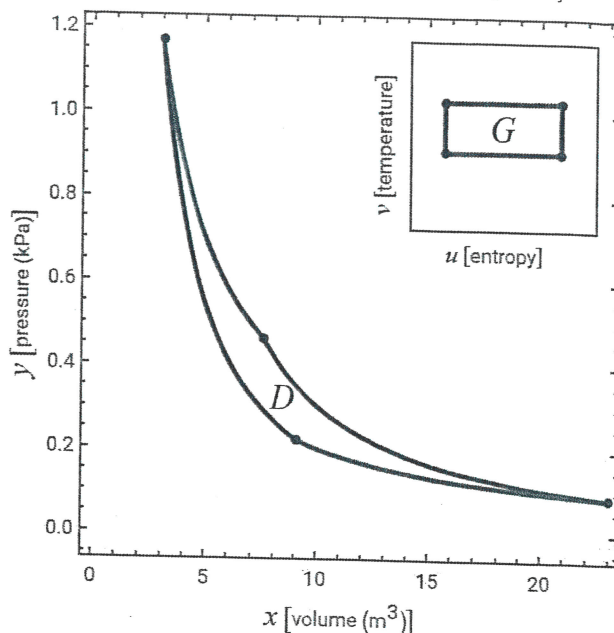
$$xy = 2, \quad xy = 7/2, \quad x^3y^2 = 12, \quad \text{and} \quad x^3y^2 = 30$$

in the xy -plane as shown in the figure.

Consider the transformation

$$u = x^3y^2 \quad \text{and} \quad v = xy$$

and let G be the region in the uv -plane corresponding to D under this transformation.



- a. Compute $\frac{\partial(u, v)}{\partial(x, y)}$ in terms of x and y .

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3x^2y^2 & 2x^3y \\ y & x \end{vmatrix} = 3x^2y^2 \cdot x - 2x^3y \cdot y = x^3y^2$$

- b. Compute $\frac{\partial(x, y)}{\partial(u, v)}$ in terms of u and v .

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{x^3y^2} = \frac{1}{u}$$

- c. Compute A by expressing it as a double integral in the uv -plane over the region G .

$$\begin{aligned} A &= \iint_D dx \, dy = \iint_G \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv = \int_2^{7/2} \int_{12}^{30} \left| \frac{1}{u} \right| du \, dv = \int_2^{7/2} \int_{12}^{30} \frac{1}{u} du \, dv \\ &= \int_2^{7/2} \left[\ln|u| \right]_{u=12}^{u=30} dv = \int_2^{7/2} (\ln(30) - \ln(12)) dv = \ln\left(\frac{5}{2}\right) \cdot v \Big|_2^{7/2} \\ &= \ln\left(\frac{5}{2}\right) \cdot \left(\frac{7}{2} - 2\right) = \frac{3}{2} \ln\left(\frac{5}{2}\right) \end{aligned}$$

$$\iint_D f(x, y) dx \, dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv \quad \text{where} \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$