5. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+2^n)(n+2^{-n})}$.

$$c_n = \frac{1}{(n+2^n)\cdot(n+2^{-n})} \Rightarrow \frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1+2^{n+1})\cdot(n+1+2^{-(n+1)})}}{\frac{1}{(n+2^n)\cdot(n+2^{-n})}}$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{(n\cdot 2^{-n}+1)\cdot(1+n^{-1}\cdot 2^{-n})}{((n+1)\cdot 2^{-(n+1)}+1)\cdot(1+n^{-1}+n^{-1}\cdot 2^{-(n+1)})} = \frac{1}{2}\cdot\frac{(0+1)\cdot(1+0)}{(0+1)\cdot(1+0+0)} = \frac{1}{2} \Rightarrow R=2$$

$\underline{x=2}$: $$\sum_{n=0}^{\infty} \frac{x^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{2^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{1}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}$$

$$c = \lim_{n\to\infty} \frac{\frac{1}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(n\cdot 2^{-n}+1)\cdot(1+n^{-1}\cdot 2^{-n})} = \frac{1}{(0+1)\cdot(1+0)} = 1$$

Since $c=1>0$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (the harmonic series),

$\sum_{n=1}^{\infty} \frac{2^n}{(n+2^n)\cdot(n+2^{-n})}$ diverges by LCT.

$\underline{x=-2}$: $$\sum_{n=0}^{\infty} \frac{x^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}$$

Let $b_n = \frac{1}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}$. Then $b_n > 0$ for $n \geq 1$ and $\lim_{n\to\infty} b_n = 0$.

Moreover $b_n > b_{n+1}$ for sufficiently large n because:

$((n+1)\cdot 2^{-(n+1)}+1)\cdot(n+1+2^{-(n+1)}) > 1\cdot(n+1) = n+1 > n+(n^2+1+n\cdot 2^{-n})\cdot 2^{-n} = (n\cdot 2^{-n}+1)\cdot(n+2^{-n})$

as $\lim_{n\to\infty}(n^2+1+n\cdot 2^{-n})\cdot 2^{-n} = 0 \Rightarrow (n^2+1+n\cdot 2^{-n})\cdot 2^{-n} < 1$ for sufficiently large n.

Hence $\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+2^n)\cdot(n+2^{-n})}$ converges by AST.

The interval of convergence of the power series is $[-2, 2)$.

[In this problem you may use the fact that $\lim_{n\to\infty} n^k 2^{-n} = 0$ for all constants k.]