

3a. Find  $f(1/4)$  if  $f$  is a continuous function on  $(-1/2, 1/2)$  satisfying:

$$\int_0^{x^2} f(t/x) dt = x^2 \tan(\pi x) \quad \text{for } 0 < |x| < 1/2$$

$$\int_0^{x^2} f(t/x) dt = x \int_0^x f(u) du \Rightarrow x \int_0^x f(u) du = x^2 \tan(\pi x) \quad \text{for } |x| < \frac{1}{2}$$

$u = t/x$   
 $du = dt/x$

(FTC1)

$$\Rightarrow \int_0^x f(u) du = x \tan(\pi x) \Rightarrow f(x) = \tan(\pi x) + x \sec^2(\pi x) \cdot \pi$$

for  $|x| < \frac{1}{2}$  for  $|x| < \frac{1}{2}$

$$\Rightarrow f(1/4) = \tan\left(\frac{\pi}{4}\right) + \frac{1}{4} \sec^2\left(\frac{\pi}{4}\right) \cdot \pi = 1 + \frac{1}{4} \cdot (\sqrt{2})^2 \cdot \pi = 1 + \frac{\pi}{2}$$

3b. Suppose that  $g$  is a differentiable function on the interval  $(-10, 10)$  such that the slope of the normal line at each point  $(x, y)$  of the graph  $y = g(x)$  is equal to  $y$ . Find  $g(8)$  if  $g(0) = 5$ .

$$g(x) = (\text{slope of normal line}) = -\frac{1}{g'(x)} \Rightarrow y = -\frac{1}{dy/dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{y}$$

$$\Rightarrow y dy = -dx \Rightarrow \int y dy = \int -dx \Rightarrow \frac{1}{2} y^2 = -x + C$$

$$(x, y) = (0, 5)$$

$$\Rightarrow \frac{1}{2} \cdot 5^2 = -0 + C \Rightarrow C = \frac{25}{2}$$

$$y^2 = 25 - 2x$$

Hence  $g(x)^2 = 25 - 2x$  for  $-10 < x < 10$ .

$$g \text{ is differentiable on } (-10, 10) \Rightarrow g \text{ is continuous on } (-10, 10)$$

$$g(x)^2 = 25 - 2x \geq 5 \text{ for } -10 < x < 10 \Rightarrow g(x) \neq 0 \text{ on } (-10, 10)$$

$$\Rightarrow \text{By IVT, } g(x) > 0 \text{ for all } -10 < x < 10 \text{ or } g(x) < 0 \text{ for all } -10 < x < 10$$

Since  $g(0) = 5 > 0$ , we have  $g(x) > 0$  for all  $-10 < x < 10$ .

Hence  $g(x) = \sqrt{25 - 2x}$  for  $-10 < x < 10 \Rightarrow g(8) = 3$