

3. Suppose that h is a continuous function on $(-\infty, \infty)$ satisfying $f(x) = g(x)h(x)$ for all x in $(-\infty, \infty)$ where:

$$f(x) = \begin{cases} \sin(\pi/x) & \text{if } x > 1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ \pi(3 - \sqrt{5-4x}) & \text{if } x < -1 \end{cases} \quad \left| \quad g(x) = \begin{cases} 1 - 1/x & \text{if } x > 1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ x^2 - 1 & \text{if } x < -1 \end{cases}$$

a. Show that there is a point c such that $h(c) = 0$.

$$h(2) = \frac{f(2)}{g(2)} = \frac{\sin(\pi/2)}{1 - \frac{1}{2}} = \frac{1}{1/2} = 2 > 0$$

$$h(-5) = \frac{f(-5)}{g(-5)} = \frac{\pi(3 - \sqrt{25})}{(-5)^2 - 1} = -\frac{\pi}{12} < 0$$

Since h is continuous on $[-5, 2]$, by IVT there is c in $(-5, 2)$ such that $h(c) = 0$.

b. Find $h(-1)$.

$$h(-1) = \lim_{x \rightarrow -1^-} h(x) = \lim_{x \rightarrow -1^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow -1^-} \frac{\pi(3 - \sqrt{5-4x})}{x^2 - 1} = \lim_{x \rightarrow -1^-} \frac{\pi(3 - \sqrt{5-4x})(3 + \sqrt{5-4x})}{(x^2 - 1)(3 + \sqrt{5-4x})}$$

$$= \lim_{x \rightarrow -1^-} \frac{\pi(9 - (5-4x))}{(x^2 - 1)(3 + \sqrt{5-4x})} = \lim_{x \rightarrow -1^-} \frac{\pi \cdot 4 \cdot (1+x)}{(x-1)(x+1)(3 + \sqrt{5-4x})}$$

$$= \lim_{x \rightarrow -1^-} \frac{4\pi}{(x-1)(3 + \sqrt{5-4x})} = \frac{4\pi}{(-1-1)(3 + \sqrt{9})} = -\frac{\pi}{3}$$

c. Find $h(1)$.

$$h(1) = \lim_{x \rightarrow 1^+} h(x) = \lim_{x \rightarrow 1^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 1^+} \frac{\sin(\pi/x)}{1 - \frac{1}{x}} = \pi \lim_{t \rightarrow 0^+} \frac{\sin(\pi - t)}{t}$$

$t = \pi(1 - \frac{1}{x})$

$$= \pi \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = \pi \cdot 1 = \pi$$

d. Suppose that h is also differentiable on $(-\infty, \infty)$. Show that there is a point d such that $h'(d) = \frac{2\pi}{3}$.

Since h is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$,

by MVT there is d in $(-1, 1)$ such that:

$$h'(d) = \frac{h(1) - h(-1)}{1 - (-1)} = \frac{\pi - (-\frac{\pi}{3})}{2} = \frac{2\pi}{3}$$