

4a. Suppose that f is differentiable on $(-\infty, \infty)$ and the slope of the tangent line at every point (x, y) on the graph of $y = f(x)$ is equal to x/y . Find $f(2)$ if $f(1) = 2$.

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y \, dy = x \, dx \Rightarrow \int y \, dy = \int x \, dx \Rightarrow \frac{1}{2} y^2 = \frac{1}{2} x^2 + C'$$

$$\xRightarrow{(x,y)=(1,2)} \frac{1}{2} \cdot 2^2 = \frac{1}{2} \cdot 1^2 + C' \Rightarrow C' = \frac{3}{2}$$

$$\text{Hence } f(x)^2 = x^2 + 3 \text{ for all } x.$$

$$\left. \begin{array}{l} f \text{ is differentiable on } (-\infty, \infty) \Rightarrow f \text{ is continuous on } (-\infty, \infty) \\ f(x)^2 = x^2 + 3 \geq 3 \text{ for all } x \Rightarrow f(x) \neq 0 \text{ for all } x \end{array} \right\}$$

$$\Rightarrow \text{By IVT, } f(x) > 0 \text{ for all } x \text{ or } f(x) < 0 \text{ for all } x$$

$$\text{Since } f(1) = 2 > 0, \text{ we have } f(x) > 0 \text{ for all } x$$

$$\text{Hence } f(x) = \sqrt{x^2 + 3} \text{ for all } x. \text{ Therefore } f(2) = \sqrt{7}.$$

4b. Suppose that a continuous function f satisfies

$$\int_0^x f(t) \, dt + \int_0^{\sqrt{x}} f(t^2) \, dt + \int_0^{\sqrt[3]{x}} f(t^3) \, dt = x$$

for all $x \geq 0$. Find $f(64)$.

$$\Downarrow \frac{d}{dx}$$

$$f(x) + f((\sqrt{x})^2) \frac{d\sqrt{x}}{dx} + f((\sqrt[3]{x})^3) \frac{d\sqrt[3]{x}}{dx} = 1 \quad \text{by Leibniz Rule}$$

$$\Downarrow$$

$$f(x) \cdot \left(1 + \frac{1}{2} x^{-1/2} + \frac{1}{3} x^{-2/3}\right) = 1$$

$$\Downarrow x=64$$

$$f(64) \cdot \left(1 + \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{16}\right) = 1$$

$$\Downarrow$$

$$f(64) = \frac{12}{13}$$